Investment in time preference and long-run distribution

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Ramsey's conjecture

- Go back to the first-year graduate macro, consider the problem of maximizing

$$\int_0^\infty v(c(t))e^{-\beta t}dt,$$

subject to

$$\dot{k} = rk - c$$

- Set up the Hamiltonian as

$$H = v(c)e^{-\beta t} + \lambda(rk - c)$$

- The FOCs are

$$\begin{aligned} 0 &= \frac{\partial H}{\partial c} = v'(c)e^{-\beta t} - \lambda \\ \dot{\lambda} &= -\frac{\partial H}{\partial k} = -r\lambda \end{aligned}$$

- From the FOCs, we have

$$\begin{aligned} 0 &= v''(c)e^{-\beta t}\dot{c} - \beta v'(c)e^{-\beta t} - \dot{\lambda} \\ &= v''(c)e^{-\beta t}\dot{c} - \beta v'(c)e^{-\beta t} + rv'(c)e^{-\beta t} \end{aligned}$$

  implying

$$\frac{\dot{c}}{c} = \frac{(\beta - r)v'(c)}{cv''(c)}$$

  Note that v'/cv'' is a negative constant when v is a power function.

- Generically, we have to have either

$$\beta > r$$

  or

$$\beta < r$$

- When r is adjusted to marginal return to capital in the long-run, it is expected to satisfy

$$\beta = r,$$

but this can never be met when different households have different βs.

- The long-run steady state condition in interior can be met only for the most patient household, and the others "perish."

- This is so-called Ramsey's conjecture, explaining an emergence of "classes."
- Ramsey (1923)
- Becker (1980), Bewley (1982) — Discrete time
- Mitra and Sorger (2013) — Continuous time

Two kinds of “uneasiness”

1. Such long-run state does not look normatively right, although there is nothing wrong in terms of classical concept of welfare.
   — Especially under intergenerational interpretation.

2. It is unrealistic.

Point 2 motivates a number of studies on endogenous time preference.

Endogenous time preference models

- Letting discount rate depend on some variable x, either a decision variable (such as consumption) or a state variable (capital or habit), in the form

$$\beta(x)$$

  where x follows certain law of motion.

- The standard discounted utility

$$U = \int_0^\infty v(c(t))e^{-\beta t}dt$$

- With endogenous discount rate, it is

$$U = \int_0^\infty v(c(t))e^{-\int_0^t \beta(x(\tau))d\tau}dt$$

Or, put differently,

- The standard discounted utility

$$U_s = \int_s^\infty v(c(t))e^{-\beta(t-s)}dt$$

which is in the recursive form

$$\dot{U}_s = -v(c(s)) + \beta U_s$$

- With endogenous discount rate, it is

$$U_s = \int_s^\infty v(c(t))e^{-\int_s^t \beta(x(\tau))d\tau}dt$$

which is in the recursive form

$$\dot{U}_s = -v(c(s)) + \beta(x(s))U_s$$

- In the long-run the variable x is flexibly adjusted so as to meet

$$\beta(x) = r$$

- They (mostly) show this is stable.

- Uzawa (1968), Epstein (1983), Lucas and Stokey (1984), Hirose and Ikeda (2008): $\beta(c)$, where c is current consumption

- Koopmans (1960), Iwai (1972), Epstein (1987, 1987), Benhabib, Jafarey and Nishimura (1988) — More general recursive utilities

- The above classes are stationary. Time preference depends only on current consumption. But is it really "Fisherian?"

- Shi and Epstein (1993): $\beta(h)$, increasing in h, where h is habit capital. — Establishing patience is indirectly costly.

- Consider that a household can make **materially costly** investment in "patience capital" π.

- Can be viewed as an extension of Becker and Mulligan (1997), Doepke and Zilibotti (2005,2008).

- $\beta(\pi)$ decreasing in π

- The law of motion

$$\dot{\pi} = -\delta\pi + g(z)$$

  where z denotes investment

- Intergenerational interpretation: cultural inheritance

- After Doepke and Zilibotti (2005,2008), there is a growing attention to investment in patience capital in the context of endogenous formation of "spirit of capitalism."

- Kawagishi (2014), Haruyama and Park (2017)

- The implication of investment in patience capital to long-run distribution of consumption and wealth in relation to the rather classic problem is not known.

- Does investment in time preference lead to survival and coexistence of heterogenous types, or opposite?

- We show that the interior steady state is unstable, in the sense that the dimension of stable manifold is one, seen in the two-dimensional space of state variables (π, k).

The preference model

- Continuous time, infinite horizon

- Two goods at each moment, one is for pure consumption, the other for "investment consumption"

- Denote such path by $(c, z) = (c(t), z(t))_{t \geq 0}$

- Preference represented in the form

$$\begin{aligned} U(c, z) &= \int_0^\infty v(c(t)) e^{-\int_0^t \beta(\pi(\tau)) d\tau} dt \\ \dot{\pi}(t) &= -\delta \pi(t) + g(z(t)) \\ \pi(0) &= \text{given} \end{aligned}$$

where $\pi = (\pi(t))_{t \geq 0}$ denotes the path of patience capital

Assumptions

Assumption 1 $v : \mathbb{R}_+ \to [0, \overline{v})$ is twice-continuously differentiable on $\mathbb{R}_{++}$ and satisfies $v' > 0$ and $v'' < 0$. Also, it satisfies $\lim_{c \to 0} v'(c) = \infty$ and $\lim_{c \to \infty} v'(c) = 0$.

Assumption 2 $\beta : \mathbb{R}_{++} \to (\underline{\beta}, \infty)$ is twice-continuously differentiable on $\mathbb{R}_{++}$, and satisfies $\beta' < 0$ and $\beta'' > 0$. Also, it holds $\lim_{\pi \to 0} \beta(\pi) = \infty$ and $\lim_{\pi \to \infty} \beta(\pi) = \underline{\beta}$.

Assumption 3 $g : \mathbb{R}_+ \to \mathbb{R}_+$ is twice-continuously differentiable on $\mathbb{R}_{++}$, and satisfies $g' > 0$, $g'' < 0$, and $g(0) = 0$. Also, it satisfies $\lim_{z \to 0} g'(z) = \infty$ and $\lim_{z \to \infty} g'(z) = 0$.

Dynamic competitive equilibrium

- One input good and labour. No preference for leisure

- Production function with constant returns to scale,

  $$F(K, L)$$

  satisfying

  $$F(aK, aL) = aF(K, L)$$

- Depreciation of capital is already taken into account.

- Output good is either consumed, held for capital accumulation or used in patience capital production. Assume linear conversion with rate 1:1, for simplicity.

Definition 1 An allocation (c, z, k) is a dynamic competitive equilibrium if there exists a path $(r, w) = (r(t), w(t))_{t \geq 0}$ such that (c_i, z_i, k_i) maximizes

$$U_i(c_i, z_i) = \int_0^\infty v_i(c_i(t)) e^{-\int_0^t \beta_i(\pi_i(\tau)) d\tau} dt$$

under the constraints

$$\begin{aligned} \dot{\pi}_i(t) &= -\delta_i \pi_i(t) + g_i(z_i(t)) \\ \dot{k}_i(t) &= r(t) k_i(t) + w(t) - c_i(t) - z_i(t) \\ c_i(t), z_i(t) &\geq 0 \\ k_i(0) &= \text{given} \end{aligned}$$

for every $i \in I$, so that the No-Ponzi condition

$$\lim_{t \to \infty} \frac{k_i(t)}{e^{\int_0^t r(\tau) d\tau}} \geq 0$$

is met, and $(\sum_{i \in I} k_i(t), n)$ solves the profit maximization problem

$$\max_{K,L} F(K, L) - r(t)K - w(t)L$$

and the market-clearing condition

$$\sum_{i \in I} c_i(t) + \sum_{i \in I} z_i(t) + \sum_{i \in I} \dot{k}_i(t) \;=\; F\left(\sum_{i \in I} k_i(t), n\right)$$

is met for all t.

- Note that the maximized profit is zero in equilibrium, at every moment, because of constant returns to scale.

Under the No-Ponzi condition, we can consolidate the series of sequential budget constraints into one, as

$$\int_0^\infty \frac{c_i(t) + z_i(t)}{e^{\int_0^t r(\tau)d\tau}} dt \ \leq \ \int_0^\infty \frac{w(t)}{e^{\int_0^t r(\tau)d\tau}} dt + r(0)k_i(0)$$

Proposition 1 If $(c_i, z_i, k_i)_{i \in I}$ is a dynamic competitive equilibrium allocation then $(c_i, z_i)_{i \in I}$ is an Arrow-Debreu-McKenzie equilibrium allocation.

Proposition 2 Arrow-Debreu-McKenzie equilibrium allocation (c, z) is Paerto-efficient. It is also ex-ante envy-free if $k_i(0) = k_j(0)$ for all $i, j \in I$.

Let

$$\alpha_i(t) = e^{-\int_0^t \beta_i(\pi_i(\tau))d\tau}$$

which follows the differential equation

$$\dot{\alpha}_i(t) = -\beta_i(\pi_i(t))\alpha_i(t)$$

Then the maximization problem for a generic price-taking household is formulated as

$$\max \int_0^\infty \alpha_i v_i(c_i) dt$$

subject to

$$\begin{aligned}
\dot{k}_i &= rk_i + w - c_i - z_i \\
\dot{\pi}_i &= -\delta_i \pi_i + g_i(z_i) \\
\dot{\alpha}_i &= -\beta_i(\pi_i)\alpha_i \\
c_i, z_i &\geq 0 \\
k_i(0) &= \text{given} \\
\pi_i(0) &= \text{given}
\end{aligned}$$

Set up the Hamiltonian as

$$H_i = \alpha_i v_i(c_i) + \lambda_i \left[rk_i + w - c_i - z_i \right] + \mu_i[-\delta_i \pi_i + g_i(z_i)] + \nu_i[-\beta_i(\pi_i)\alpha_i]$$

Then the individually optimal path is characterized by

$$\begin{aligned}
\alpha_i v_i'(c_i) - \lambda_i &= 0 \\
-\lambda_i + \mu_i g_i'(z_i) &= 0 \\
\dot{\lambda}_i &= -\lambda_i r \\
\dot{\mu}_i &= \delta_i \mu_i + \beta_i'(\pi_i)\nu_i \alpha_i \\
\dot{\nu}_i &= -v_i(c_i) + \beta_i(\pi_i)\nu_i \\
\dot{k}_i &= rk_i + w - c_i - z_i \\
\dot{\pi}_i &= -\delta_i \pi_i + g_i(z_i) \\
\dot{\alpha}_i &= -\beta_i(\pi_i)\alpha_i
\end{aligned}$$

plus

three transversality conditions

$$\begin{aligned}\lim_{t\to\infty} \lambda_i(t)k_i(t) &\leq 0\\ \lim_{t\to\infty} \mu_i(t)\pi_i(t) &\leq 0\\ \lim_{t\to\infty} \nu_i(t)\alpha_i(t) &\leq 0\end{aligned}$$

Together with the No-Ponzi condition the first transversality condition reduces to

$$\lim_{t\to\infty} \frac{k_i(t)}{e^{\int_0^t r(\tau)d\tau}} = 0.$$

Also, because of positivity of π_i, together with the first-order condition the second one reduces to

$$\lim_{t\to\infty} \frac{\pi_i(t)}{g_i'(z_i(t))e^{\int_0^t r(\tau)d\tau}} = 0,$$

and it is relevant only when π_i diverges to positive infinity. The third one is met by any path under the bounded utility assumption.

We eliminate λ_i, μ_i and α_i, while ν_i cannot be, and obtain the dynamics

$$
\begin{aligned}
\dot{c}_i &= \frac{(\beta_i(\pi_i) - r)v_i'(c_i)}{v_i''(c_i)} \\
\dot{z}_i &= -\frac{g_i'(z_i)}{g_i''(z_i)}\left[r + \delta_i + \frac{g_i'(z_i)\beta_i'(\pi_i)\nu_i}{v_i'(c_i)}\right] \\
\dot{\nu}_i &= -v_i(c_i) + \beta_i(\pi_i)\nu_i \\
\dot{\pi}_i &= -\delta_i\pi_i + g_i(z_i) \\
\dot{k}_i &= rk_i + w - c_i - z_i
\end{aligned}
$$

On the other hand, from the profit-maximization condition in equilibrium it holds

$$
\begin{aligned}
r &= F_1\left(\sum_i k_i, n\right) \\
w &= F_2\left(\sum_i k_i, n\right)
\end{aligned}
$$

By taking the time derivative of the above, we obtain the dynamics of equilibrium interest rate and wage

$$
\begin{aligned}
\dot{r} &= F_{11}\left(\sum_i k_i, n\right)\left[r\sum_i k_i + nw - \sum_i c_i - \sum_i z_i\right] \\
\dot{w} &= F_{21}\left(\sum_i k_i, n\right)\left[r\sum_i k_i + nw - \sum_i c_i - \sum_i z_i\right]
\end{aligned}
$$

Interior steady state $((c_i^*, z_i^*, \nu_i^*, \pi_i^*, k_i^*)_{i\in I}, r^*, w^*)$, if exists, is determined by

$$
\begin{aligned}
\beta_i(\pi_i^*) &= r^* \\
g_i(z_i^*) &= \delta_i \pi_i^* \\
\frac{v_i'(c_i^*)}{v_i(c_i^*)} &= -\frac{g_i'(z_i^*)\beta_i'(\pi_i^*)}{r^*(r^* + \delta_i)} \\
\nu_i^* &= \frac{v_i(c_i^*)}{r^*} \\
k_i^* &= \frac{c_i^* + z_i^* - w^*}{r^*} \\
r^* &= F_1\left(\sum_i k_i^*, n\right) \\
w^* &= F_2\left(\sum_i k_i^*, n\right)
\end{aligned}
$$

The Jacobian matrix evaluated at the interior steady state takes the form of $(5n+2) \times (5n+2)$ matrix

$$\begin{pmatrix} A_1 & O & \cdots & O & P_1 \\ O & A_2 & \cdots & O & P_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \cdots & A_n & P_n \\ E & E & \cdots & E & Q \end{pmatrix}$$

where

$$A_i = \begin{pmatrix} 0 & 0 & 0 & \frac{\beta_i' v_i'}{v_i''} & 0 \\ -(r+\delta_i)\frac{g_i' v_i''}{g_i'' v_i'} & r+\delta_i & (r+\delta_i) r \frac{g_i'}{g_i'' v_i} & (r+\delta_i)\frac{g_i' \beta_i''}{g_i'' \beta_i'} & 0 \\ -v_i' & 0 & r & \frac{\beta_i' v_i}{r} & 0 \\ 0 & g_i' & 0 & -\delta_i & 0 \\ -1 & -1 & 0 & 0 & r \end{pmatrix}$$

$$P_i = \begin{pmatrix} -\frac{v_i'}{v_i''} & 0 \\ -\frac{g_i'}{g_i''} & 0 \\ 0 & 0 \\ 0 & 0 \\ k_i & 1 \end{pmatrix}$$

for each i and

$$
E = \begin{pmatrix} -F_{11} & -F_{11} & 0 & 0 & F_{111}\left[r\sum_i k_i + nw - \sum_i c_i - \sum_i z_i\right] + F_{11}r \\ -F_{21} & -F_{21} & 0 & 0 & F_{211}\left[r\sum_i k_i + nw - \sum_i c_i - \sum_i z_i\right] + F_{21}r \end{pmatrix}
$$

$$
Q = \begin{pmatrix} F_{11}\sum_i k_i & F_{11}n \\ F_{21}\sum_i k_i & F_{21}n \end{pmatrix}
$$

The case of linear technology

- Assume $F(K,L) = rK + wL$, and $r > \max_i \underline{\beta}_i$.

- It suffices to look at each household problem separately.

- We only need to look at the stability property of each diagonal block A_i separately.

- Since π_i and k_i are the state variables the number of stable roots required for stability with a unique optimal path is exactly two.

$$A_i = \begin{pmatrix} 0 & 0 & 0 & \frac{\beta_i' v_i'}{v_i''} & 0 \\ -(r+\delta_i)\frac{g_i' v_i''}{g_i'' v_i'} & r+\delta_i & (r+\delta_i) r \frac{g_i'}{g_i'' v_i} & (r+\delta_i)\frac{g_i' \beta_i''}{g_i'' \beta_i'} & 0 \\ -v_i' & 0 & r & \frac{\beta_i' v_i}{r} & 0 \\ 0 & g_i' & 0 & -\delta_i & 0 \\ -1 & -1 & 0 & 0 & r \end{pmatrix}$$

Proposition 3 The number of stable roots for each A_i is one. Let θ_{i1} denote the only stable root for A_i, and $\theta_{i2}, \theta_{i3}, \theta_{i4}, \theta_{i5}$ denote the unstable roots, ordered in the ascending manner according to their real parts, then at least $\theta_{i1}, \theta_{i4}, \theta_{i5}$ are real and it holds

$$\theta_{i1} < -\delta_i < 0 < \mathrm{Re}\theta_{i2} \leq \mathrm{Re}\theta_{i3} < \theta_{i4} = r < r + \delta_i < \theta_{i5}.$$

Proof

Pick any i. Since it is clear that one eigenvalue of A_i is r, we can restrict attention to its submatrix

$$\begin{pmatrix} 0 & 0 & 0 & \frac{\beta' v_i'}{v_i''} \\ -(r+\delta_i)\frac{g_i' v_i''}{g_i'' v_i'} & r+\delta_i & (r+\delta_i) r \frac{g_i'}{g_i'' v_i} & (r+\delta_i)\frac{g_i' \beta_i''}{g_i'' \beta_i'} \\ -v_i' & 0 & r & \frac{\beta_i' v_i}{r} \\ 0 & g_i' & 0 & -\delta_i \end{pmatrix}$$

Then its characteristic polynomial is

$$
\begin{aligned}
p_i(\theta_i) &= (\theta_i + \delta_i)\theta_i(\theta_i - r)(\theta_i - (r + \delta_i)) \\
&\quad -(r + \delta_i)\frac{(g_i')^2}{g_i''}\frac{\beta_i''}{\beta_i'}\theta_i(\theta_i - r) \\
&\quad -(r + \delta_i)r\frac{(g_i')^2}{g_i''}\beta_i'\left(1 - \frac{(v_i')^2}{v_i''v_i}\right)
\end{aligned}
$$

which is an even function around $r/2$.

One can verify $-(r + \delta_i)r\frac{(g_i')^2}{g_i''}\beta_i'\left(1 - \frac{(v_i')^2}{v_i''v_i}\right) < 0$.

The rest follows from high-school mathematics.

- One-dimensional stable manifold, a curve
- The eigenvector for $\theta_{i4} = r$ is $(0,0,0,0,1)^T$, and that for each θ_{ik}, $k = 1,2,3,5$, is

$$\begin{pmatrix} \frac{1}{\theta_{ik}} \frac{\beta_i' v_i'}{v_i''} \\ \frac{\theta_{ik}+\delta_i}{g_i'} \\ \frac{1}{r-\theta_{ik}} \left[\frac{1}{\theta_{ik}} \frac{\beta_i' (v_i')^2}{v_i''} - \frac{\beta_i' v_i}{r} \right] \\ 1 \\ \frac{1}{r-\theta_{ik}} \left[\frac{1}{\theta_{ik}} \frac{\beta_i' v_i'}{v_i''} + \frac{\theta_{ik}+\delta_i}{g_i'} \right] \end{pmatrix}$$

- The local dynamics is given by

$$\begin{pmatrix} c_i - c_i^* \\ z_i - z_i^* \\ \nu_i - \nu_i^* \\ \pi_i - \pi_i^* \\ k_i - k_i^* \end{pmatrix} = s_{i4} e^{rt} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$+ \sum_{k=1,2,3,5} s_{ik} e^{\theta_{ik} t} \begin{pmatrix} \frac{1}{\theta_k} \frac{\beta_i' v_i'}{v_i''} \\ \frac{\theta_k + \delta}{g_i'} \\ \frac{1}{r-\theta_k} \left[\frac{1}{\theta_k} \frac{\beta_i' (v_i')^2}{v_i''} - \frac{\beta_i' v_i}{r} \right] \\ 1 \\ \frac{1}{r-\theta_k} \left[\frac{1}{\theta_k} \frac{\beta_i' v_i'}{v_i''} + \frac{\theta_k + \delta_i}{g_i'} \right] \end{pmatrix}$$

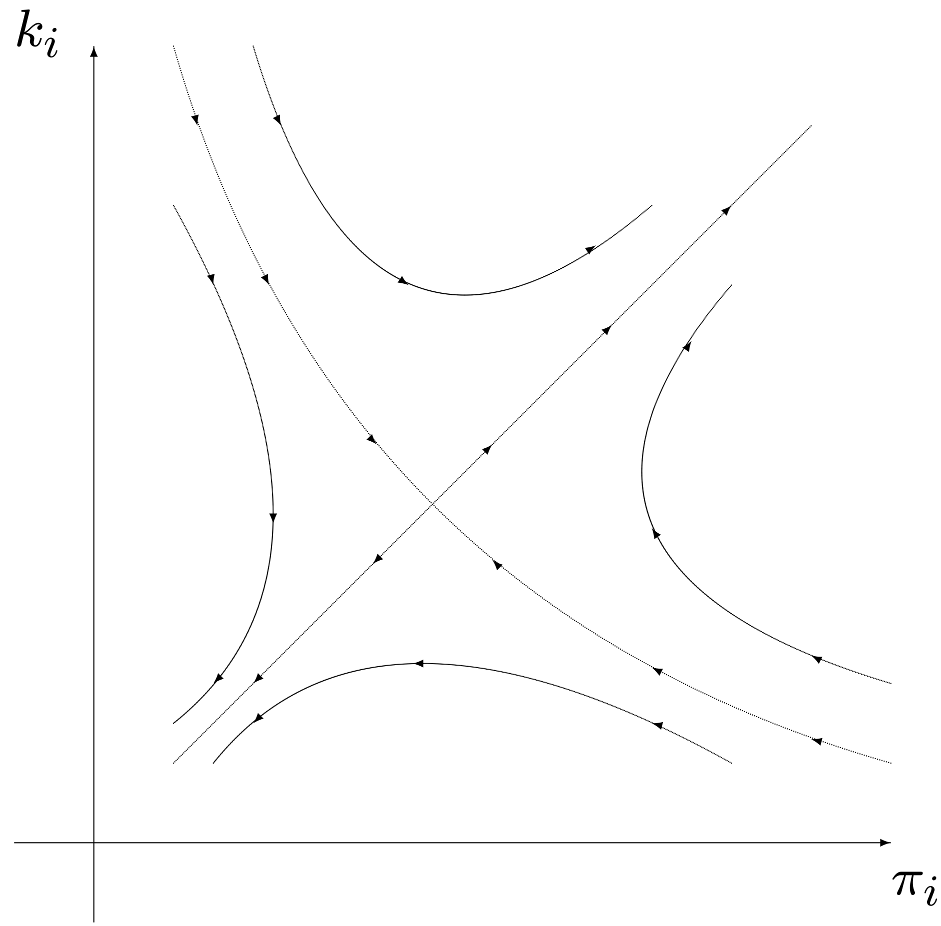
where the constants $\{s_{ik}\}_{k=1,2,3,4,5}$ are determined according to the initial values.

Note that the eigenvectors projected on the space of state variables (π_i, k_i) is

- downward for θ_{i1} and θ_{i5}
- upward for θ_{i2} and θ_{i3} when they are real, and
- vertical for $\theta_{i4} = r$.

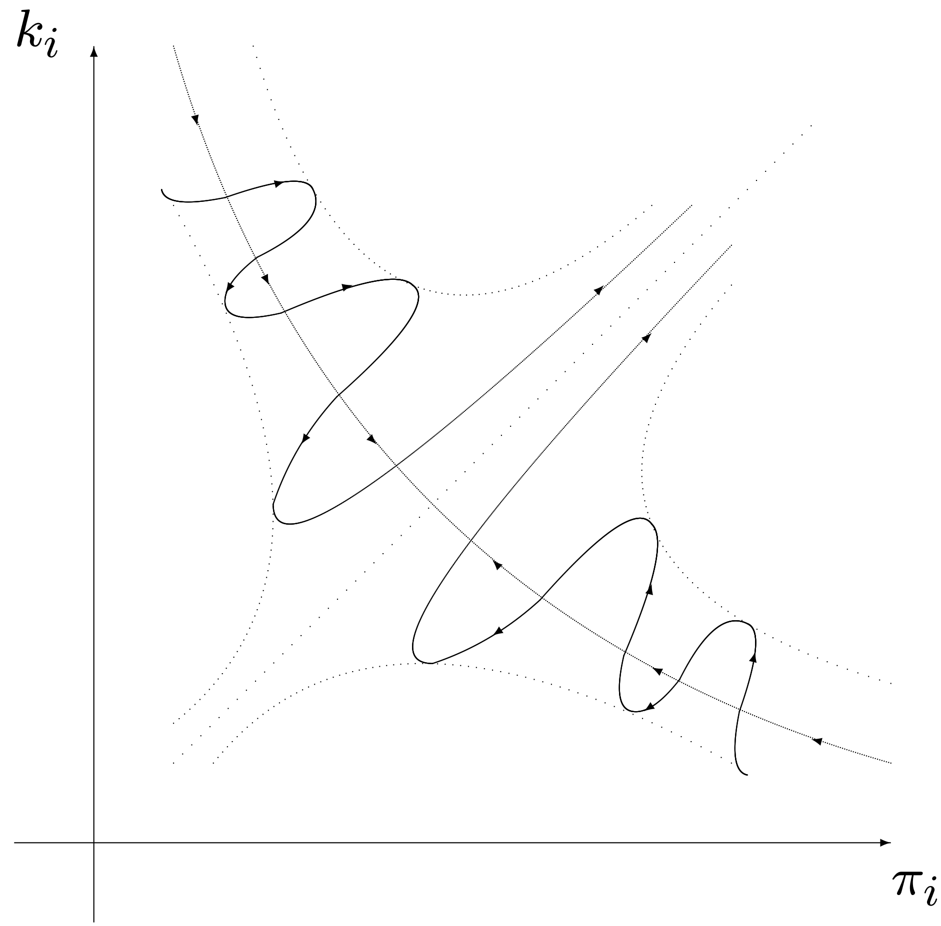
Hence the curve of stable manifold projected on the space of state variables is downward-sloping.

Since $\theta_{i4} = r$ and $\theta_{i5} > r + \delta_i$ are inconsistent with the conjunction of No-Ponzi condition and transversality condition, the associated upper-left/lower-right directions won't play a role in the global dynamics.



k_i
π_i

- Saddle point **in the space of state variables**
- Dependence on initial value of (π_i, k_i)
- Self-cofirming nature of classes and their dispositions


k_i
π_i

Diminishing returns to capital and capital/labour complementarity

- We can take a class of CRS production functions which coincides with $rK + wL$ along the ray connecting the origin and $\left(\sum_i k_i^*, n\right)$.

- We can make F_{11} and F_{21} arbitrarily close to zero at $\left(\sum_i k_i^*, n\right)$.

- Then the characteristic polynomial for the Jacobian matrix evaluated at the steady state takes the form

$$\left(\theta - F_{11} \sum_i k_i\right)(\theta - F_{21} n) \prod_i |\theta I - A_i| + F_{11} F_{21} G(\theta)$$

  where I denotes the 5×5 identity matrix and $G(\theta)$ is a $5n+1$-th order polynomial.

Since F_{11} and F_{21} are arbitrarily close to zero, the set of roots in the characteristic equation is arbitrarily close to the one in

$$\left(\theta - F_{11} \sum_i k_i\right) (\theta - F_{21} n) \prod_i |\theta I - A_i| = 0$$

in which the number of stable roots is $n + 1$, as $F_{11} < 0$ and $F_{21} > 0$.

Summing up, we obtain the following claim.

Proposition 4 Pick any $r > \max_i \underline{\beta}_i$ and w for which an interior steady state $(c_i^*, z_i^*, \nu_i^*, \pi_i^*, k_i^*)_{i \in I}$ exists in the corresponding linear technology economy. Then there is a range of technology F with constant returns to scale and $F_{11}\left(\sum_i k_i^*, n\right) < 0$ and $F_{21}\left(\sum_i k_i^*, n\right) > 0$, which results in the same interior steady state with $F_1\left(\sum_i k_i^*, n\right) = r$ and $F_2\left(\sum_i k_i^*, n\right) = w$ such that the number of stable roots in the linearized system is $n + 1$.

Note that when $n = 1$ the above result says that the corresponding single-household optimal growth problem has a unique and stable steady state, while instability shows up again in the competitive economy with $n > 1$.

Conclusions

- A complete and global characterization of equilibrium path is obviously desired, but...

- A numerical approach may help in order to get a picture, as one can still use the recursive method.

$$V(\pi, k) = \max \left\{ v(c) + \frac{1}{1+\beta(\pi)} V(\pi', k') \right\}$$

where

$$\begin{aligned} \pi' &= (1-\delta)\pi + g(z) \\ k' &= (1+r)k - c - z \end{aligned}$$

which is solvable by the simple contraction-mapping method.

- One can think of a more general form of production of patience capital, like

$$\dot{\pi} = h(z, \pi)$$

- Welfare implications. Ex-post redistributions?

- We considered an entirely frictionless economy with perfect foresights. Does any market friction or bounded foresight strengthen instability or rather stabilize long-run distribution?
  — Sorger (2008): Imperfect competition with standard DU